Cálculo de integrales reales definidas con teo. de los residuos

$$\int_{-\pi}^{\pi} F(\cos\theta, \operatorname{sen}\theta)\, d\theta \longrightarrow \int_C F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{iz}\, dz$$

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$\frac{dz}{iz} = d\theta$

C: circ. $|z|=1$. positivamente orientado.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\operatorname{sen}\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$F(\cos\theta, \operatorname{sen}\theta)$ continua para $\theta \in [-\pi, \pi]$ $\Rightarrow$ $F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$ no tiene singularidades en C (circunf. unitaria)

Ejemplo: $\displaystyle\int_{-\pi}^{\pi} \frac{1}{(a+b\cos\theta)^2}\, d\theta$ con $0<b<a$

$\cos\theta = \frac{z+z^{-1}}{2}$ siendo $z = e^{i\theta}$

$$\int_{-\pi}^{\pi} \frac{1}{(a+b\cos\theta)^2}\, d\theta = \int_C \frac{1}{\left[a+b\left(\frac{z+z^{-1}}{2}\right)\right]^2}\frac{dz}{iz} = \int_C \frac{4}{\left(2a+bz+b\frac{1}{z}\right)^2}\frac{dz}{iz}$$

$C: |z|=1$

$$\int_C \frac{-4iz}{(2az+bz^2+b)^2}\, dz$$

Singularidades:

$bz^2 + 2az + b = 0$

$z^2 + 2\frac{a}{b}z + 1 = 0$

$$z = \frac{-2\frac{a}{b} \pm \sqrt{4\left(\frac{a}{b}\right)^2 - 4}}{2} = -\frac{a}{b} \pm \sqrt{\underbrace{\left(\frac{a}{b}\right)^2 - 1}_{>0}}$$

Sea $\lambda = \frac{a}{b} > 1$ $\quad z_1 = -\lambda - \sqrt{\lambda^2-1}$ $\quad z_1 \in RI(C)$?

$z_2 = -\lambda + \sqrt{\lambda^2-1}$ $\quad z_2 \in RI(C)$?

$\lambda = \frac{a}{b} > 1 \Rightarrow z_1 = -\lambda - \sqrt{\lambda^2-1} < -\lambda < -1$

$\Rightarrow |z_1| > 1 \notin Rl(c)$

Por otro lado:

$\sqrt{\lambda^2-1} < \lambda \quad \Rightarrow |z_2| = |-\lambda + \sqrt{\lambda^2-1}| = \lambda - \sqrt{\lambda^2-1}$

$\lambda - 1 > 0$

$$\lambda - 1 < \lambda + 1$$
$$(\lambda-1)^2 < (\lambda+1)(\lambda-1)$$
$$|\lambda - 1| < \sqrt{\lambda^2-1}$$
$$\lambda - 1 < \sqrt{\lambda^2-1}$$
$$\underbrace{\lambda - \sqrt{\lambda^2-1}}_{|z_2|} < 1$$

$|z_2| < 1 \quad \Rightarrow z_2 \in Rl(c)$

Otra forma:

$$(z-z_1)(z-z_2) = z^2 - (z_1+z_2)z + z_1 z_2 = z^2 + 2\lambda z + 1 = 0$$

$\Rightarrow z_1 . z_2 = 1$

$|z_1| = \frac{1}{|z_2|} \qquad$ si $|z_1| > 1 \Rightarrow |z_2| < 1 \Rightarrow z_2 \in Rl(c)$

$$\int_{-\pi}^{\pi} \frac{1}{(a+b\cos\theta)^2}\, d\theta = \int_c \frac{-4iz}{(bz^2+2az+b)^2}\, dz = \int_c \underbrace{\frac{-4iz}{b^2(z-z_1)^2(z-z_2)^2}}_{f(z)}\, dz$$

$= 2\pi i \operatorname{Res}(f, z_2)$

z_2 polo orden 2 de $f(z)$.

$$\operatorname{Res}(f, z_2) = \lim_{z\to z_2} \left[(z-z_2)^2 f(z)\right]' = \lim_{z\to z_2} \left(\frac{-4iz}{b^2(z-z_1)^2}\right)' =$$

$$\lim_{z\to z_2} \frac{-4i\cdot(z-z_1)^2 b^2 + 4iz\, b^2\, 2(z-z_1)}{b^4(z-z_1)^4} = \lim_{z\to z_2} \frac{-4ib^2(z-z_1)\left[z-z_1-2z\right]}{b^4(z-z_1)^4}$$

$$= \lim_{z\to z_2} \frac{-4i}{b^2} \frac{\left[-z_1-z\right]}{(z-z_1)^3} = +\frac{4i}{b^2}\frac{(z_1+z_2)}{(z_2-z_1)^3} = \frac{4i}{b^2}\left(-2\frac{a}{b}\right)\cdot \frac{1}{\left(2\sqrt{\left(\frac{a}{b}\right)^2-1}\right)^3}$$

$$= \frac{-8ia}{b^3}\,\frac{1}{8\left(\frac{\sqrt{a^2-b^2}}{b^3}\right)^3} = \frac{-ia}{\left(\sqrt{a^2-b^2}\right)^3} = \frac{-ia}{(a^2-b^2)^{3/2}}$$

$$\int_{-\pi}^{\pi}\frac{1}{(a+b\cos\theta)^2}\,d\theta = 2\pi i\left(\frac{-ia}{\left(\sqrt{a^2-b^2}\right)^3}\right) = \frac{2\pi a}{\left(\sqrt{a^2-b^2}\right)^3}$$

$$\int_0^{\pi} ---- = \frac{1}{2}\int_{-\pi}^{\pi} ----$$

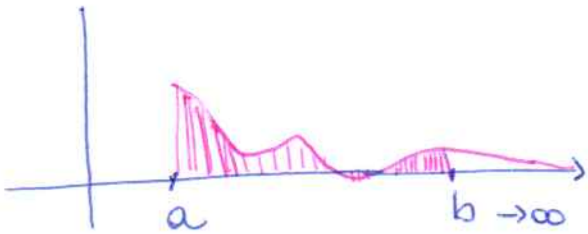
Integrales impropias.

- intervalo no acotado
- función no acotada.

Sea $f:[a,+\infty)\to\mathbb{R}$

Def: si existe $\int_a^b f(x)\,dx$ para todo b mayor a a

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$



Si el límite existe (es finito):

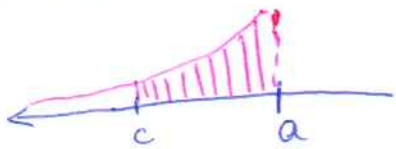
"$\int_a^{\infty} f(x)\,dx$ converge (C)"

y/o "$f(x)$ es integrable en $[a,\infty)$"

Sea $f:(-\infty,a]\to\mathbb{R}$

Def: si existe $\int_c^a f(x)\,dx$ para todo c menor a a

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{c\to-\infty}\int_c^a f(x)\,dx$$



Si el límite existe (es finito)

"$\int_{-\infty}^{a} f(x)\,dx$ converge (C)"

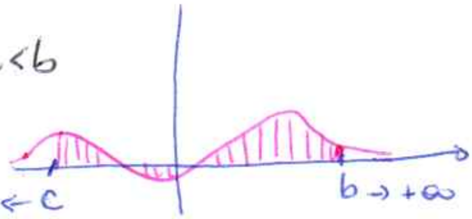
y/o "$f(x)$ es integrable en $[-\infty,a]$"

NO ES CORRECTO: "f converge"

Sea $f: \mathbb{R} \to \mathbb{R}$

<u>Def</u>: si existen $\int_c^b f(x)dx$ para todos c y b, $c<b$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{b \to +\infty \\ c \to -\infty}} \int_c^b f(x)dx$$



Si existe el límite: "$\int_{-\infty}^{\infty} f(x)dx$ converge (C)" ó "f es integrable en $\mathbb{R}$ o en $(-\infty, +\infty)$"

<u>Valor principal</u>: $VP \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x)dx$

Si $\int_{-\infty}^{\infty} f(x)dx$ converge $\Rightarrow$ existe su VP y $VP\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

<u>Convergencia absoluta</u>

Si $\int_{\alpha}^{\beta} |f(x)|\,dx$ converge (C), se dice que $\int_{\alpha}^{\beta} f(x)dx$ converge absolutamente. (C.A)

(α finito o $\alpha = -\infty$, β finito o $\beta = +\infty$)

<u>Proposición</u>: Si $\int_{\alpha}^{\beta} f(x)dx$ C.A. $\Rightarrow$ C.

<u>Ejemplos</u>

1. $\int_0^{\infty} \frac{1}{1+x^2}dx = \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2}dx = \lim_{b \to \infty} \operatorname{arctg}(b) - \operatorname{arctg}(0) = \frac{\pi}{2}$ (C)

2. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim_{\substack{b \to +\infty \\ c \to -\infty}} \int_c^b \frac{1}{1+x^2}dx = \lim_{\substack{b \to +\infty \\ c \to -\infty}} \operatorname{arctg}(b) - \operatorname{arctg}(c) = \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) = \pi$ (C)

3. $\int_0^{\infty} \operatorname{sen} x\,dx = \lim_{b \to +\infty} \int_0^b \operatorname{sen} x\,dx = \lim_{b \to +\infty} -\cos(b) + \cos(0)\ \nexists$ no C.

4. $\int_0^{\infty} k\,dx = \lim_{b \to +\infty} kb - k.0 = \infty$ no C.

(5) $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^p}\,dx \underset{\text{si } p\neq 1}{=} \lim_{b\to+\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \lim_{b\to\infty} \frac{b^{1-p}-1}{1-p}$

$$= \begin{cases} \dfrac{1}{p-1} & \text{si } p>1 \\ \infty & \text{si } p<1 \end{cases}$$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b - \ln 1 = \infty$$

$\Rightarrow$ $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ converge si y sólo si $p>1$

$\frac{1}{x^p}$ es integrable en $[1,\infty)$ si y sólo si $p>1$

(6) $\displaystyle VP\int_{-\infty}^{\infty} \operatorname{sen} x\,dx = \lim_{R\to+\infty} \int_{-R}^{R} \operatorname{sen} x\,dx = \lim_{R\to+\infty} -\cos(R)+\cos(-R) = 0$

mientras que $\displaystyle\int_{-\infty}^{\infty} \operatorname{sen} x\,dx = \lim_{\substack{b\to+\infty\\ c\to-\infty}} \int_c^b \operatorname{sen} x\,dx = \underbrace{\lim_{\substack{b\to+\infty\\ c\to-\infty}} -\cos(b)+\cos(c)}_{\nexists}$

sen(x) no es integrable en $\mathbb{R}$

Integral impropia con funciones no acotadas.

Sea $f:(a,b] \to \mathbb{R}$

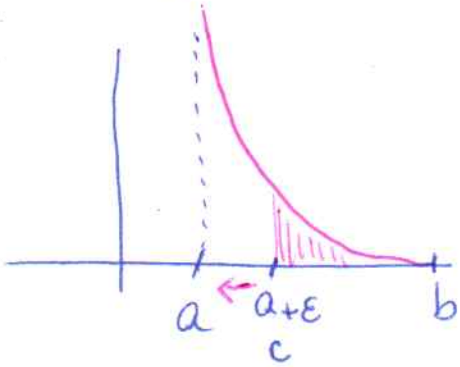
<u>Def</u>: si existe $\displaystyle\int_{a+\varepsilon}^{b} f(x)\,dx$ para todo $\varepsilon>0$:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

si existe el límite (es finito)

"$\displaystyle\int_a^b f(x)\,dx$ converge"

y/o $f(x)$ es integrable en $(a,b]$



Similarmente:

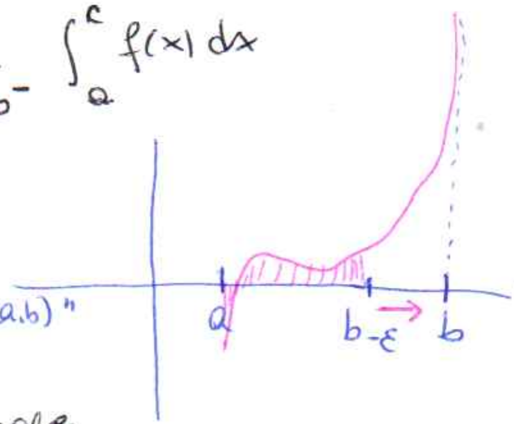
Def Sea $f:[a,b) \to \mathbb{R}$

si existe $\int_a^{b-\varepsilon} f(x)\,dx$ para todo $\varepsilon > 0$:

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

si existe y es finito.

"$\int_a^b f(x)\,dx$ converge" y/o "f es integrable en $[a,b)$"



Ejercicio: dar las definiciones para integrales impropias con problemas de no acotación

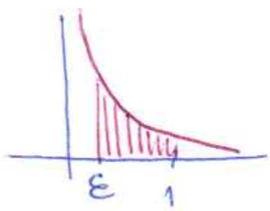
- dentro del intervalo
- en ambos extremos
- en varios puntos dentro del intervalo.

dar def de valor principal para estas int. imp.

Ejemplo.

① $$\int_0^1 \frac{1}{x^p}\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x^p}\,dx \underset{\text{si } p \neq 1}{=} \lim_{\varepsilon \to 0^+} \left.\frac{x^{-p+1}}{1-p}\right|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} \frac{1-\varepsilon^{1-p}}{1-p}$$

$$= \begin{cases} \frac{1}{1-p} & \text{si } p<1 \\ \infty & \text{si } p>1 \end{cases}$$



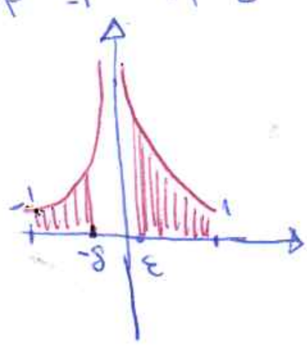
$p=1$: $\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0^+} \ln 1 - \ln \varepsilon \quad \nexists$

$\frac{1}{x^p}$ es integrable en $(0,1]$ si y sólo si $p<1$

$\int_0^1 \frac{1}{x^p}dx$ converge si y sólo si $p<1$

② $$\int_{-1}^1 \frac{1}{|x|^p}\,dx = \lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} \int_{-1}^{-\delta} \frac{1}{|x|^p}\,dx + \int_\varepsilon^1 \frac{1}{x^p}\,dx \overset{p \neq 1}{=} \lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} \left.\frac{-|x|^{1-p}}{1-p}\right|_{-1}^{-\delta} + \left.\frac{x^{1-p}}{1-p}\right|_\varepsilon^1$$

$$= \lim_{\substack{\varepsilon \to 0^+ \\ \delta \to 0^+}} \frac{-\delta^{1-p} + 1 + 1 - \varepsilon^{1-p}}{1-p} = \begin{cases} \frac{2}{1-p} & \text{si } p<1 \\ \infty & \text{si } p>1 \end{cases}$$



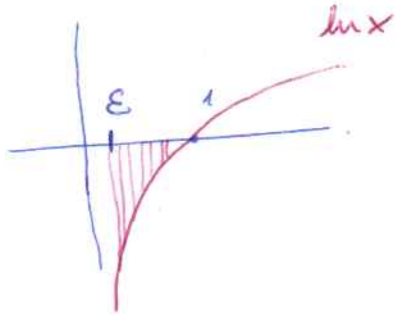
Si $p=1$, no converge.

3 $\displaystyle\int_0^1 \ln(x)\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \ln(x)\,dx = \lim_{\varepsilon\to 0^+} x(\ln(x)-1)\Big|_\varepsilon^1 =$

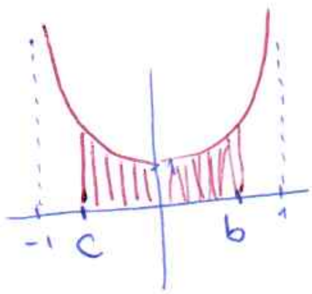
$$= \lim_{\varepsilon\to 0^+} -1 - \varepsilon(\ln(\varepsilon)-1) = -1.$$

"$\ln x$ es integrable en $(0,1]$"

$\int_0^1 \ln x\,dx$ converge.

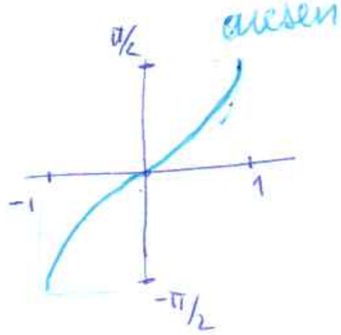


4 $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{b\to 1^- \\ c\to -1^+}} \int_c^b \frac{1}{\sqrt{1-x^2}}\,dx =$



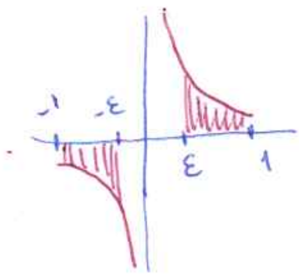
$$= \lim_{\substack{b\to 1^- \\ c\to -1^+}} \operatorname{arcsen}(b) - \operatorname{arcsen}(c) =$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



5 $\displaystyle VP\int_{-1}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+} \int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_\varepsilon^1 \frac{1}{x}\,dx =$

$$= \lim_{\varepsilon\to 0^+} \ln|-\varepsilon| - \ln|-1| + \ln 1 - \ln \varepsilon = \lim_{\varepsilon\to 0} 0 = 0$$



I I

Converge?

a qué número converge?

~~Con~~ Si se puede aplicar definición: se responden ambas.

si no ...

Criterios de convergencia de I.I.

Ⓐ COMPARACION

Si $0 \le f(x) \le g(x)$ para $x \in D$ (D: intervalo)

y $\int_D g(x)dx$ converge $\Rightarrow$ $\int_D f(x)\,dx$ converge

Ⓑ COMPARACION AL LIMITE

Sean $0 \le f(x)$, $0 < g(x)$. Sea $l = \lim_{x \to b^-} \frac{f(x)}{g(x)}$ (b finito o $b = +\infty$)

Si l es finito, $l \neq 0 \Rightarrow \int_a^b f(x)dx$ converge $\Leftrightarrow \int_a^b g(x)dx$ converge

Ⓒ CRITERIO DE DIRICHLET-ABEL

Sean: f monótona ~~decreciente~~, $\lim_{x \to b^-} f(x) = 0$ (b finito o $b = +\infty$)

g una función tal que $\left| \int_a^c g(x)dx \right| < M$, para algún M, para todo $c < b$

$\Rightarrow \int_a^b f(x) \cdot g(x)\,dx$ converge.

Estudio de convergencia. Ejemplos

(I) $$\boxed{\int_0^\infty \underbrace{\frac{\operatorname{sen}(bx)}{1+x^2}}\,dx = I}$$

es acotada en intervalos $[0,b]$, la integral es impropia porque ~~está~~ el intervalo es no acotado

Veamos convergencia ~~[illegible]~~ absoluta:

$$0 \le \left|\frac{\operatorname{sen}(bx)}{1+x^2}\right| \le \frac{1}{1+x^2} \quad \text{para todo } x.$$

$\frac{1}{1+x^2}$ es integrable en $[0,\infty)$ $\Rightarrow$ $\left|\frac{\operatorname{sen}(bx)}{1+x^2}\right|$ es integrable en $[0,\infty)$

o:

$$\int_0^\infty \frac{1}{1+x^2}dx \text{ converge} \Rightarrow \int_0^\infty \left|\frac{\operatorname{sen}(bx)}{1+x^2}\right| dx \text{ converge.}$$

I converge absolutamente $\Rightarrow$ converge.

(I') $$\boxed{\int_{-\infty}^0 \frac{\operatorname{sen}(bx)}{1+x^2}\,dx} \underset{x=-t}{=} \int_\infty^0 \frac{\operatorname{sen}(-bt)}{1+(-t)^2}(-dt) = \int_0^\infty -\frac{\operatorname{sen}(bt)}{1+t^2}\,dt = -I$$

$$\Rightarrow \int_{-\infty}^0 \frac{\operatorname{sen}(bx)}{1+x^2}\,dx \text{ converge.}$$

(I'') $\boxed{\int_{-\infty}^\infty \frac{\operatorname{sen}(bx)}{1+x^2}\,dx}$ converge, ya que

$$\int_{-\infty}^\infty \frac{\operatorname{sen}(bx)}{1+x^2}\,dx = \lim_{\substack{b\to+\infty\\ c\to-\infty}} \int_c^b \frac{\operatorname{sen}(bx)}{1+x^2}\,dx = \lim_{\substack{b\to+\infty\\ c\to-\infty}} \int_c^0 \cdots + \int_0^b \cdots$$

$$= \int_{-\infty}^0 \cdots + \int_0^\infty \cdots = -I + I = 0$$

(II) $\boxed{\int_0^{\infty} \underbrace{\frac{1}{\sqrt{x}(1+x)}} dx}$

no acotada en entorno de 0, y el intervalo es no acotado.

Analiza: $\underbrace{\int_0^1 \frac{1}{\sqrt{x}(1+x)} dx}_{I_1}$ y $\underbrace{\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx}_{I_2}$

I_1: Sea $f(x) = \frac{1}{\sqrt{x}(1+x)}$ y $g_1(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to 0^+} \frac{f(x)}{g_1(x)} = \lim_{x \to 0^+} \frac{1}{1+x} = 1.$$

Como $\int_0^1 g_1(x)dx = \int_0^1 \frac{1}{\sqrt{x}} dx = \int_0^1 \frac{1}{x^p} dx$ converge $\Rightarrow \int_0^1 f(x)dx$ converge.

$p = 1/2 < 1$

Existe I_1.

I_2: Sea $g_2(x) = \frac{1}{x^{3/2}}$

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{3/2}}{(1+x)\sqrt{x}} = \lim_{x\to\infty} \frac{x}{1+x} = 1$$

Como $\int_1^{\infty} g_2(x) dx = \int_1^{\infty} \frac{1}{x^{3/2}} dx$ converge $\Rightarrow \int_1^{\infty} f(x)dx$ converge

$p = 3/2 > 1$

COMPARACION AL LIMITE

Existe I_2.

$\Rightarrow \int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$ converge.

(III) $\boxed{\int_0^1 \underbrace{\frac{\cos x}{x}} dx}$

no acotado en $[0,1]$.

Sean $g(x) = \frac{1}{x}$, $f(x) = \frac{\cos x}{x}$.

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\cos x / x}{1/x} = \lim_{x\to 0^+} \cos x = 1.$$

Como $\int_0^1 \frac{1}{x} dx$ no converge $\Rightarrow \int_0^1 \frac{\cos x}{x} dx$ no converge.

POR COMPARACION AL LÍMITE

(IV) $\boxed{\int_0^1 \frac{\operatorname{sen} x}{x}\,dx}$ no es impropio, el integrando es acotado.

(V) $\boxed{\int_1^\infty \frac{\operatorname{sen} x}{x}\,dx.}$

Sean $f(x) = \frac{1}{x}$ y $g(x) = \operatorname{sen} x$.

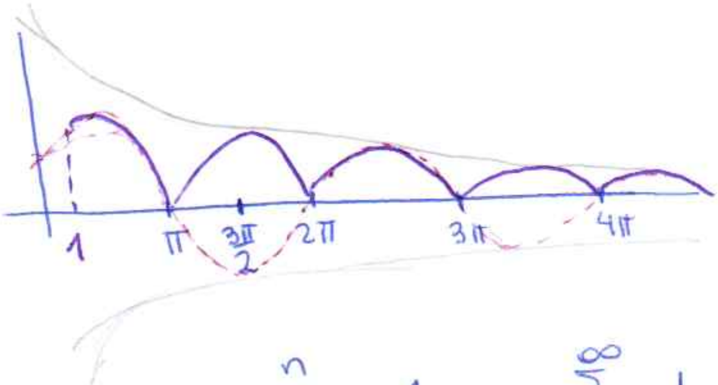
f monótona, $\lim_{x \to \infty} f(x) = 0$, y $\left| \int_1^c \operatorname{sen} x\,dx \right| = \left| -\cos c + \cos 1 \right| \leq 2$ para todo c

POR DIRICHLET-ABEL

$\Rightarrow \int_1^\infty f(x) \cdot g(x)\,dx = \int_1^\infty \frac{\operatorname{sen} x}{x}\,dx$ converge.

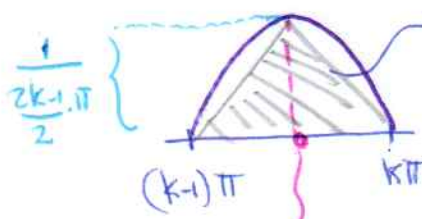
Converge absolutamente?

$$\int_\pi^\infty \left| \frac{\operatorname{sen} x}{x} \right| dx =$$



$$\lim_{n \to \infty} \int_\pi^{n\pi} \left| \frac{\operatorname{sen} x}{x} \right| dx =$$

$$\lim_{n \to \infty} \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \left| \frac{\operatorname{sen} x}{x} \right| dx \geq \lim_{n \to \infty} \sum_{k=2}^{n} \frac{1}{2k-1} = \underbrace{\sum_{k=2}^{\infty} \frac{1}{2k-1}}_{\text{comparación con } \sum \frac{1}{k}} = \infty$$



área del triángulo: $\frac{1}{2} \pi \cdot \frac{1}{\frac{(2k-1)}{2}\pi} = \frac{1}{(2k-1)}$

$(k-1)\pi + \frac{\pi}{2} = \frac{2k-1}{2}\pi$

$\Rightarrow \int_1^\infty \frac{\operatorname{sen} x}{x}\,dx$ no converge absolutamente.

(VI) $\boxed{\int_{-\infty}^{\infty} e^{-a|x|}\,dx \quad , a>0}$

$$\int_{-\infty}^{\infty} e^{-a|x|}dx = \lim_{\substack{b\to+\infty\\ c\to-\infty}} \int_c^0 e^{-a|x|}dx + \int_0^b e^{-a|x|}dx =$$

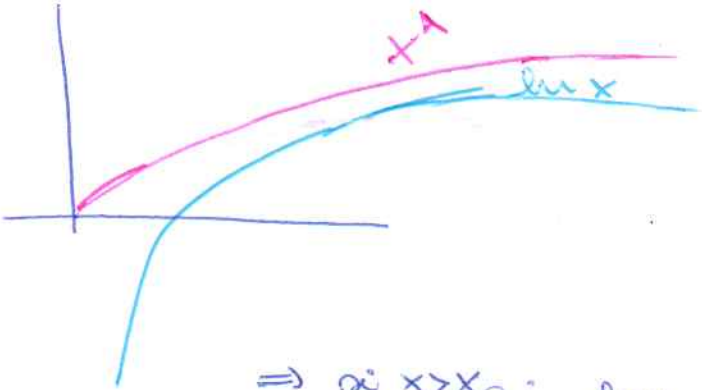
$$= \lim_{\substack{b\to+\infty\\ c\to-\infty}} \int_c^0 e^{ax}dx + \int_0^b e^{-ax}dx = \lim_{\substack{b\to+\infty\\ c\to-\infty}} \frac{1-e^{ac}}{a} + \frac{e^{-ab}-1}{-a} =$$

$= \frac{2}{a}$. $\Rightarrow$ converge!

(VII) $\boxed{\int_{1}^{\infty} \frac{\ln x}{x^2+1}\,dx}$

Sea $0<\lambda<1$

$$\lim_{x\to+\infty} \frac{\ln x}{x^\lambda} = \lim \frac{1/x}{\lambda x^{\lambda-1}} = \lim \frac{1}{\lambda x^\lambda} = 0$$



$\Rightarrow$ para algún x_0 se verifica que

$\frac{\ln x}{x^\lambda} < 1 \Rightarrow \ln x < x^\lambda$ si $x>x_0$

$$\Rightarrow \text{si } x>x_0:\ 0 \leq \frac{\ln x}{x^2+1} < \underbrace{\frac{x^\lambda}{x^2+1}}_{\varphi(x)} \approx \frac{x^\lambda}{x^2} = \frac{1}{x^{2-\lambda}}$$

Sea $g(x) = \frac{1}{x^{2-\lambda}}$

$$\lim_{x\to\infty} \frac{\varphi(x)}{g(x)} = \lim \frac{x^\lambda}{(x^2+1)} \cdot x^{2-\lambda} = 1.$$

Como $\int_{x_0}^{\infty} \frac{1}{x^{2-\lambda}}dx$ converge (porque $2-\lambda>1$ al ser $\lambda<1$)

$\Rightarrow \int_{x_0}^{\infty} \varphi(x)\,dx$ converge (comparación al límite)

$\Rightarrow \int_{x_0}^{\infty} \frac{\ln x}{x^2+1}dx$ converge (comparación)

$\Rightarrow \int_{1}^{\infty} \frac{\ln x}{x^2+1}dx$ converge.

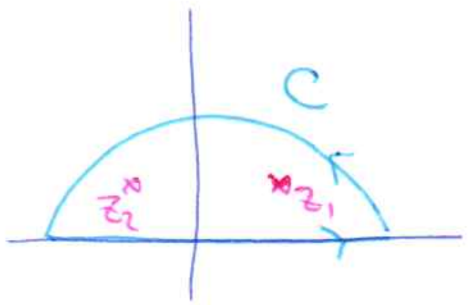
Cálculo de integrales impropias.

(A) $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{1}{1+x^4}}_{\text{acotada en } [a,b],\ \text{para todos } a,b} dx$

Por comparación con $\frac{1}{x^4}$, la integral converge.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = VP\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx$$

Sea $f(z) = \frac{1}{1+z^4}$, y C: semicirc $|z|=R$ y, $\operatorname{Im} z \geq 0$, y segmento $[-R,R]$, eje real



$$\int_C f(z)\,dz = 2\pi i\left(\sum \operatorname{Res}(f, z_j)\right)$$

Sing: $z^4 = -1$

$$e^{4i\theta} = e^{-i\pi} \Rightarrow \theta = \frac{-\pi + 2k\pi}{4}$$

$z_0 = e^{-\frac{\pi}{4}i}$, $z_1 = e^{\frac{\pi}{4}i}$, $z_2 = e^{\frac{3\pi}{4}i}$ $z_3 = e^{\frac{5\pi}{4}i}$

$$\underbrace{\int_C f(z)\,dz} = 2\pi i\left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right) \underset{\substack{\downarrow \\ \text{clase } 11/6}}{=} \pi\frac{\sqrt{2}}{2}$$

$$\int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} \underbrace{\frac{1}{1+x^2}\,dx}_{\substack{z=x \\ dz=dx}} = \pi\frac{\sqrt{2}}{2} \quad *$$

Γ_R semicirc.

$$\left|\int_{\Gamma_R} \frac{1}{1+z^4}\,dz\right| \leq \frac{1}{R^4-1}\cdot \pi R \xrightarrow{R\to\infty} 0$$

Tomando límite en $*$: $\displaystyle VP\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx = \pi\frac{\sqrt{2}}{2}$

Similarmente se aplica el procedimiento p/ integrar funciones racionales $\frac{P(x)}{Q(x)}$ con $gr(P) \leq gr(Q)+2$ y Q no tiene raíces reales.